Math 17B
Vogler
Eigenvalues and Eigenvectors for Two-By-Two Matrices

DEFINITION : Assume that A is a two-by-two matrix and X is a nonzero vector $\left(X \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)$. If

$$AX = \lambda X \ ,$$

then we say X is an *eigenvector* of A and λ is its *eigenvalue* .

FACT : If X is an eigenvector for A, then any multiple of X, say cX, is also an eigenvector since $A(cX) = cAX = c\lambda X = \lambda(cX)$.

HOW TO FIND EIGENVALUES AND EIGENVECTORS

If X is a nonzero solution to $AX = \lambda X$ then $AX = \lambda IX \quad \longrightarrow$

$$AX - \lambda IX = O \quad \longrightarrow$$

$$(A - \lambda I)X = O \quad \longrightarrow$$

$$det(A - \lambda I) = 0 \ .$$

(NOTE : If $det(A - \lambda I) \neq 0$, then matrix $A - \lambda I$ is invertible. This would imply that the only solution to $(A - \lambda I)X = O$ would be $X = O$, contradicting the fact that $X \neq O$ since X is an eigenvector.)

EXAMPLE : Find eigenvalues and eigenvectors for each matrix.

1.) $A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$, then

$$A - \lambda I = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

$$= \begin{pmatrix} -\lambda & 1 \\ -2 & -3 - \lambda \end{pmatrix} \quad \longrightarrow$$

$$det(A - \lambda I) = det \begin{pmatrix} -\lambda & 1 \\ -2 & -3 - \lambda \end{pmatrix}$$

$$= (-\lambda)(-3 - \lambda) - (1)(-2)$$

$$= \lambda^2 + 3\lambda + 2$$

$$= (\lambda + 2)(\lambda + 1) = 0 \quad \longrightarrow \quad \text{eigenvalues for } A \text{ are } \lambda = -2 \text{ and } \lambda = -1 \ .$$

Now find an eigenvector for each eigenvalue by solving $(A - \lambda I)X = O$ for X :

For $\lambda = -2$: $\left(\begin{array}{cc|c} 2 & 1 & 0 \\ -2 & -1 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \longrightarrow$

$2x_1 + x_2 = 0$ so let $x_1 = t$ any number, then $x_2 = -2x_1 = -2t$ and

$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} t \\ -2t \end{pmatrix} = t\begin{pmatrix} 1 \\ -2 \end{pmatrix}$, so choose $V_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ as an eigenvector for $\lambda = -2$.

For $\lambda = -1$: $\left(\begin{array}{cc|c} 1 & 1 & 0 \\ -2 & -2 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \longrightarrow$

$x_1 + x_2 = 0$ so let $x_2 = t$ any number, then $x_1 = -x_2 = -t$ and

$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -t \\ t \end{pmatrix} = t\begin{pmatrix} -1 \\ 1 \end{pmatrix}$, so choose $V_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ as an eigenvector for $\lambda = -1$.