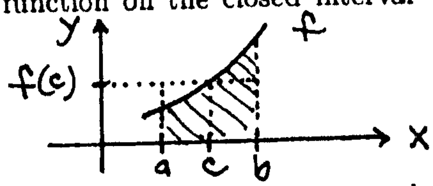
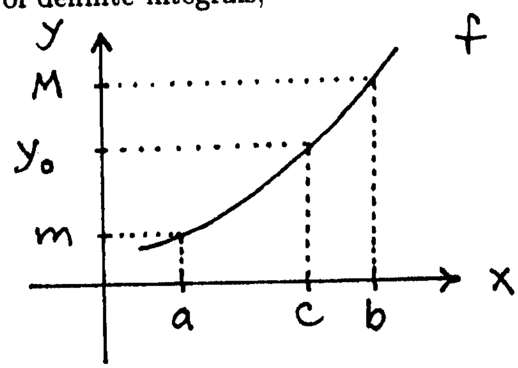
Math 17B
Vogler
Mean Value Theorem and Fundamental Theorem of Calculus

Mean Value Theorem for Integrals : If f is a continuous function on the closed interval $[a, b]$, then there is at least one number c , $a \le c \le b$, so that



$$f(c)(b-a) = \int_a^b f(x)\,dx .$$

Proof : Since f is a continuous function on the closed interval $[a, b]$, by the Maximum- and Minimum-Value Theorems, f has a maximum vaule M and a minimum value m on $[a, b]$, i.e., $m \le f(x) \le M$ on $[a, b]$. By property 8.)(p. 289) of definite integrals,



$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a) ,$$

so that

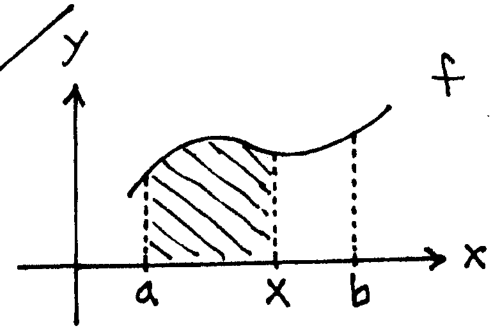
$$m \le \underbrace{\frac{1}{b-a}\int_a^b f(x)\,dx}_{\text{call this number } y_o} \le M ,$$

By the Intermediate Value Theorem (p. 119) there is at least one number c , $a \le c \le b$, so that

$$f(c) = y_o, \text{ i.e., } f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx ,$$

so that

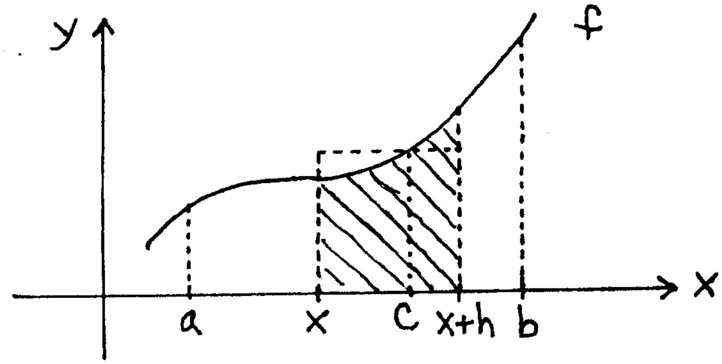
$$f(c)(b-a) = \int_a^b f(x)\,dx .$$



First Fundamental Theorem of Calculus (FTC1) : Assume that f is a continuous function on the closed interval $[a, b]$ and that $F(x) = \int_a^x f(t)\,dt$. Then $F'(x) = f(x)$.

Proof : Consider $F(x) = \int_a^x f(t)\,dt$ as the area under the graph of f above the interval $[a, x]$. Then $F(x+h)$ is the area under the graph of f above the interval $[a, x+h]$ and $F(x+h) - F(x)$ is the area of the " thin strip " from x to $x+h$, i.e., $F(x+h) - F(x) = \int_x^{x+h} f(t)\,dt$. By the Mean Value Theorem for integrals there is at least one number c , $x \le c \le x+h$, so that

$$f(c) \cdot h = \int_x^{x+h} f(t)\,dt$$

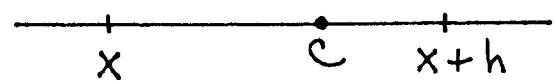


The derivative of $F(x)$ can now be computed as

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$

$$= \lim_{h \to 0} \frac{\displaystyle\int_x^{x+h} f(t)\, dt}{h}$$

$$= \lim_{h \to 0} \frac{f(c)\, h}{h}$$



$$= \lim_{h \to 0} f(c) \qquad \text{(Recall that } x \le c \le x+h.\text{)}$$

$$= f(x)\ .$$

<u>*Second Fundamental Theorem of Calculus*</u> (FTC2) : Let f be a continuous function on the closed interval $[a, b]$. Assume that $F(x)$ is an antiderivative of $f(x)$, i.e., assume that $F'(x) = f(x)$. Then

$$\int_a^b f(x)\, dx = F(x)\Big|_a^b = F(b) - F(a)\ .$$

<u>*Proof*</u> : Let $A(x) = \int_a^x f(t)\, dt$. Then $A(a) = 0$, $A(b) = \int_a^b f(t)\, dt$, and $A'(x) = f(x)$ by FTC1. But $F'(x) = f(x)$. By Corollary 2 (p. 212) to the Mean Value Theorem $F(x) = A(x) + C$ for any constant C, or

$$A(x) = F(x) - C\ .$$

Then

$$\int_a^b f(x)\, dx = \int_a^b f(t)\, dt$$

$$= A(b)$$

$$= A(b) - A(a)$$

$$= (F(b) - C) - (F(a) - C)$$

$$= F(b) - F(a)$$

$$= F(x)\Big|_a^b\ .$$