Math 17B

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Equilibria and Stability

Defn Consider the autonomous Differential Equation

$$\frac{dy}{dx} = g(y), \quad (*)$$

and assume that $\hat{y} = k$ is a constant. If $g(\hat{y}) = 0$, then we say that the constant function $\hat{y}$, as a solution to the D.E. $(*)$, is an <u>equilibrium</u> of $(*)$.

Note: To find equilibria of D.E.s of the form $(*)$, you just solve the equation $g(y) = 0$ for y algebraically.

Defn Let $\hat{y} = k$ be an equilibrium for $\frac{dy}{dx} = g(y)$, i.e $g(\hat{y}) = 0$.

1) $\hat{y}$ is <u>stable</u> if solutions to D.E. $(*)$ with nearby initial values $(t=0)$ "converge to" $\hat{y}$ as $t \to \infty$.

2) $\hat{y}$ is <u>unstable</u> if solutions to D.E. with nearby initial values $(t=0)$ "diverge away from" $\hat{y}$ as $t \to \infty$.

Classfying Stability (2 Methods)

Let $\hat{y} = k$ be an equilibrium for $\frac{dy}{dx} = g(y)$, so $g(\hat{y}) = 0$.

I) <u>Graphical Approach</u> (Using sign charts)

a) $+ \quad 0 \quad -$ $\longrightarrow g(y)$

$y = \hat{y}$ (Stable)

b) $- \quad 0 \quad +$ $\longrightarrow g(y)$

$y = \hat{y}$ (unstable)

c) $+ \quad 0 \quad +$ $\longrightarrow g(y)$

$y = \hat{y}$ (semi-stable)

d) $- \quad 0 \quad -$ $\longrightarrow g(y)$

$y = \hat{y}$ (semi-stable)

II) Analytical / Eigenvalue Approach (Using derivative)

Let $\lambda = g'(\hat{y})$, which is called the eigenvalue for $y = \hat{y}$.

Then,
a) If $\lambda < 0$, then $\hat{y}$ is stable.
b) If $\lambda > 0$, then $\hat{y}$ is unstable.
c) If $\lambda = 0$ and either $g''(\hat{y}) > 0$ or $g''(\hat{y}) < 0$, then $\hat{y}$ is semi-stable.

Ex Find and classify (stable/unstable) the equilibria for

$$\frac{dN}{dt} = N^2 - 3N - 4$$

using a) graphical approach, b) analytical approach

(Solve for roots)

$$g(N) = N^2 - 3N - 4 = (N-4)(N+1) = 0$$

$\Rightarrow \boxed{N=4}$ and $\boxed{N=-1}$ are equilibria.

a)

$$\xrightarrow{\quad + \quad 0 \quad - \quad 0 \quad + \quad} g(N)$$

$\boxed{\text{Stable}} \to N=-1$ $\qquad N=4 \leftarrow \boxed{\text{Unstable}}$

b) $g'(N) = 2N - 3$

i) $g'(-1) = 2(-1) - 3 = -5 < 0$,
so $N = -1$ is $\boxed{\text{Stable}}$

ii) $g'(4) = 2(4) - 3 = 5 > 0$,
so $N = 4$ is $\boxed{\text{Unstable}}$