Section 7.6

1.) $f(x) = e^{2x} \xrightarrow{D} f'(x) = 2e^{2x}$ and $a = 0$:

$L(x) = f(0) + f'(0)(x-0) = 1 + 2(x) \longrightarrow$

$\boxed{L(x) = 2x + 1}$

2.) $f(x) = \sin(3x) \xrightarrow{D} f'(x) = 3\cos(3x)$ and $a = 0$:

$L(x) = f(0) + f'(0)(x-a) = \sin 0 + 3\cos 0 \cdot (x-0)$

$= 0 + 3(1)x \longrightarrow \boxed{L(x) = 3x}$

3.) $f(x) = \dfrac{1}{1-x} = (1-x)^{-1} \xrightarrow{D}$

$f'(x) = -(1-x)^{-2} \cdot (-1) = (1-x)^{-2} = \dfrac{1}{(1-x)^2}$ and

$a = 0$: $L(x) = f(0) + f'(0)(x-0)$

$= 1 + 1(x) \longrightarrow \boxed{L(x) = x + 1}$

5.) $f(x) = \ln(2+x^2) \xrightarrow{D} f'(x) = \dfrac{1}{2+x^2} \cdot 2x$

and $a = 0$: $L(x) = f(0) + f'(0)(x-0)$

$= \ln 2 + (0)(x) \longrightarrow \boxed{L(x) = \ln 2}$

6.) $f(x) = \dfrac{1}{1+x} = (1+x)^{-1} \xrightarrow{D} f'(x) = -(1+x)^{-2}$

$\xrightarrow{D} f''(x) = 2(1+x)^{-3} \xrightarrow{D} f'''(x) = -6(1+x)^{-4}$

$\xrightarrow{D} f^{(4)}(x) = 24(1+x)^{-5}$ and $a = 0$:

$a_0 = f(0) = 1$, $a_1 = f'(0) = -1$,

$a_2 = \dfrac{f''(0)}{2!} = \dfrac{2}{2} = 1$, $a_3 = \dfrac{f'''(0)}{3!} = \dfrac{-6}{6} = -1$,

$a_4 = \dfrac{f^{(4)}(0)}{4!} = \dfrac{24}{24} = 1$; then

$$P_4(x) = a_0 + a_1(x-0) + a_2(x-0)^2 + a_3(x-0)^3 + a_4(x-0)^4$$
$$= 1 + (-1)x + 1\cdot x^2 + (-1)x^3 + 1\cdot x^4 \longrightarrow$$
$$\boxed{P_4 = 1 - x + x^2 - x^3 + x^4}$$

7.) $f(x) = \cos x \xrightarrow{D} f'(x) = -\sin x$
$\xrightarrow{D} f''(x) = -\cos x \xrightarrow{D} f'''(x) = \sin x$
$\xrightarrow{D} f^{(4)}(x) = \cos x \xrightarrow{D} f^{(5)}(x) = -\sin x$
and $a = 0$:

$a_0 = f(0) = \cos 0 = 1$, $a_1 = f'(0) = -\sin 0 = 0$,

$a_2 = \dfrac{f''(0)}{2!} = \dfrac{-\cos 0}{2} = -\dfrac{1}{2}$,

$a_3 = \dfrac{f'''(0)}{3!} = \dfrac{\sin 0}{6} = \dfrac{0}{6} = 0$,

$a_4 = \dfrac{f^{(4)}(0)}{4!} = \dfrac{\cos 0}{24} = \dfrac{1}{24}$,

$a_5 = \dfrac{f^{(5)}(0)}{5!} = -\dfrac{\sin 0}{120} = \dfrac{0}{120} = 0$; then

$$P_5(x) = a_0 + a_1(x-0) + a_2(x-0)^2 + a_3(x-0)^3 + a_4(x-0)^4 + a_5(x-0)^5$$
$$= 1 + (0)x + \left(-\tfrac{1}{2}\right)x^2 + (0)x^3 + \left(\tfrac{1}{24}\right)x^4 + (0)x^5 \longrightarrow$$
$$\boxed{P_5(x) = 1 - \tfrac{1}{2}x^2 + \tfrac{1}{24}x^4}$$

11.) $f(x) = \sqrt{2+x} \xrightarrow{D} f'(x) = \frac{1}{2}(2+x)^{-1/2}$
$\xrightarrow{D} f''(x) = -\frac{1}{4}(2+x)^{-3/2} \xrightarrow{D}$
$f'''(x) = \frac{3}{8}(2+x)^{-5/2}$ and $a = 0$:

$$a_0 = f(0) = \sqrt{2}, \quad a_1 = f'(0) = \frac{1}{2\sqrt{2}} = \frac{1}{2^{3/2}},$$

$$a_2 = \frac{f''(0)}{2!} = \frac{-\frac{1}{4} \cdot \frac{1}{2^{3/2}}}{2} = \frac{-1}{8} \cdot \frac{1}{2^{3/2}}$$

$$= \frac{-1}{2^3 \cdot 2^{3/2}} = \frac{-1}{2^{9/2}}, \quad a_3 = \frac{f'''(0)}{3!}$$

$$= \frac{\frac{3}{8} \cdot \frac{1}{2^{5/2}}}{6} = \frac{1}{16 \cdot 2^{5/2}} = \frac{1}{2^4 \cdot 2^{5/2}} = \frac{1}{2^{13/2}};$$

then

$$P_3(x) = a_0 + a_1(x-0) + a_2(x-0)^2 + a_3(x-0)^3 \rightarrow$$

$$\boxed{P_3(x) = \sqrt{2} + \frac{1}{2^{3/2}} x - \frac{1}{2^{9/2}} x^2 + \frac{1}{2^{13/2}} x^3};$$

$f(x) = \sqrt{2+x}$ so $f(0.1) = \sqrt{2.1} \approx 1.449137$
and $P_3(0.1) \approx 1.449138$.

15.) $f(x) = \tan x \xrightarrow{D} f'(x) = \sec^2 x \xrightarrow{D}$
$f''(x) = 2 \cdot \sec x \cdot \sec x \tan x \rightarrow$
$f''(x) = 2 \sec^2 x \tan x$ and $a = 0$;

$a_0 = f(0) = \tan 0 = 0$,

$a_1 = f'(0) = \sec^2 0 = 1^2 = 1$,

$a_2 = f''(0) = 2\sec^2 0 \cdot \tan 0 = 2(1)^2(0) = 0$;

then $P_2(x) = a_0 + a_1(x-0) + a_2(x-0)^2$
$= 0 + (1)x + (0)x^2 \rightarrow \boxed{P_2(x) = x}$;

$f(x) = \tan x$ so
$f(0.1) = \tan(0.1) \approx 0.1003$ and $P_2(0.1) = 0.1$

16.) $f(x) = \ln(1+x) \xrightarrow{D} f'(x) = \frac{1}{1+x} = (1+x)^{-1}$
$\xrightarrow{D} f''(x) = -(1+x)^{-2} \xrightarrow{D} f'''(x) = 2(1+x)^{-3}$
$\xrightarrow{D} f^{(4)}(x) = -6(1+x)^{-4}$ and $a = 0$:

$a_0 = f(0) = \ln 1 = 0$, $a_1 = f'(0) = 1$,

$a_2 = \frac{f''(0)}{2!} = \frac{-1}{2}$, $a_3 = \frac{f'''(0)}{3!} = \frac{2}{6} = \frac{1}{3}$,

; then

$$P(x) = a_0 + a_1(x-0) + a_2(x-0)^2 + a_3(x-0)^3$$
$$= 0 + (1)x + \left(\tfrac{-1}{2}\right)x^2 + \left(\tfrac{1}{3}\right)x^3$$

$$\boxed{P(x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3}$$;

$f(x) = \ln(1+x)$ so $f(0.1) = \ln 1.1 \approx 0.09531$
and $P(0.1) \approx 0.09533$

19.) $f(x) = \sqrt{x} \xrightarrow{D} f'(x) = \frac{1}{2}x^{-1/2} \xrightarrow{D}$
$f''(x) = -\frac{1}{4}x^{-3/2} \xrightarrow{D} f'''(x) = \frac{3}{8}x^{-5/2}$ and
$a = 1$:

$a_0 = f(1) = 1$, $a_1 = f'(1) = \frac{1}{2}$,

$a_2 = \frac{f''(1)}{2!} = \frac{-\frac{1}{4}}{2} = \frac{-1}{8}$, $a_3 = \frac{f'''(1)}{3!} = \frac{3/8}{6} = \frac{1}{16}$; then

$P_3(x) = a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3 \longrightarrow$

$$\boxed{P_3(x) = 1 + \tfrac{1}{2}(x-1) - \tfrac{1}{8}(x-1)^2 + \tfrac{1}{16}(x-1)^3}$$;

$f(x) = \sqrt{x}$ so $f(2) = \sqrt{2} \approx 1.4142$ and
$P_3(2) = 1.4375$

22.) $f(x) = x^{1/5} \xrightarrow{D} f'(x) = \frac{1}{5}x^{-4/5} \xrightarrow{D}$
$f''(x) = -\frac{4}{25}x^{-9/5} \xrightarrow{D} f'''(x) = \frac{36}{125}x^{-14/5}$ and

$a = -1$:

$a_0 = f(-1) = -1$, $a_1 = f'(-1) = \frac{1}{5}$,
$a_2 = \frac{f''(-1)}{2!} = \frac{4/25}{2} = \frac{2}{25}$, $a_3 = \frac{f'''(-1)}{3!} = \frac{\frac{36}{125}}{6} = \frac{6}{125}$;

then $P_3(x) = a_0 + a_1(x+1) + a_2(x+1)^2 + a_3(x+1)^3 \longrightarrow$

$$\boxed{P_3(x) = -1 + \frac{1}{5}(x+1) + \frac{2}{25}(x+1)^2 + \frac{6}{125}(x+1)^3};$$

$f(x) = x^{1/5}$ so $f(-0.9) = (-0.9)^{1/5} \approx -0.979148$
and $P_3(-0.9) = -0.979152$

27.) $f(x) = e^x$ on $[0, 2]$, $a = 0$, and
Taylor Error $|R_{n+1}(x; 0)| < 0.001$:

$f(x) = e^x \xrightarrow{D} f'(x) = e^x \xrightarrow{D} f''(x) = e^x \xrightarrow{D}$
$f'''(x) = e^x \xrightarrow{D} \cdots \xrightarrow{D} f^{(n)}(x) = e^x \xrightarrow{D}$
$\underline{f^{(n+1)}(x) = e^x}$; then for x in $[0, 2]$

Taylor Error $|R_{n+1}(x;0)| = \left| \frac{f^{(n+1)}(c)}{(n+1)!}(x-0)^{n+1} \right|$

$= \frac{e^c}{(n+1)!}|x|^{n+1} \qquad (0 \le c \le x \le 2)$

$\le \boxed{\frac{e^2}{(n+1)!} 2^{n+1} < 0.001}$; guess and check :

n	$\frac{e^2 \cdot 2^{n+1}}{(n+1)!}$
1	14.778
2	9.852
3	4.926
4	1.970
5	0.657
6	0.188
7	0.047
8	0.0104
9	0.0021
10	0.0003 ≤ 0.001

So choose $\boxed{n = 10}$ (or bigger)

28.) $f(x) = \cos x$ on $[0,1]$, $a = 0$, and Taylor Error $|R_{n+1}(x;0)| < 0.01$:

$f(x) = \cos x \xrightarrow{D} f'(x) = -\sin x \xrightarrow{D}$
$f''(x) = -\cos x \xrightarrow{D} f'''(x) = \sin x \xrightarrow{D}$
$f^{(4)}(x) = \cos x \xrightarrow{D} \cdots$

$f^{(n)}(x) = \pm \sin x$ or $\pm \cos x$;

for x in $[0,1]$ Taylor Error

$$|R_{n+1}(x;0)| = \left| \frac{f^{(n+1)}(c)}{(n+1)!} (x-0)^{n+1} \right|$$

$$= \frac{|x|^{n+1}}{(n+1)!} \left| f^{(n+1)}(c) \right| \qquad (0 \le c \le x \le 1)$$

but $|f^{(n+1)}(c)| = |\pm \sin c$ or $\pm \cos c| \leq 1$

so

$$\frac{|x|^{n+1}}{(n+1)!} |f^{(n+1)}(c)| \leq \frac{1^{n+1}}{(n+1)!} (1) =$$

$$= \boxed{\frac{1}{(n+1)!} < 0.01}$$; guess and check

n	$\frac{1}{(n+1)!}$
1	0.5
2	0.167
3	0.042
4	0.008

≤ 0.01

So choose $\boxed{n=4}$ (or bigger).

30.) $f(x) = \ln(1+x)$ on $[0, 0.1]$, $a=0$, and Taylor Error $|R_{n+1}(x;0)| < 0.001$:

$$f(x) = \ln(1+x) \xrightarrow{D} f'(x) = \frac{1}{1+x} = (1+x)^{-1} \xrightarrow{D}$$
$$f''(x) = -(1+x)^{-2} \xrightarrow{D} f'''(x) = 2(1+x)^{-3} \xrightarrow{D}$$
$$f^{(4)}(x) = -3\cdot 2(1+x)^{-4} \xrightarrow{D} f^{(5)}(x) = 4\cdot 3\cdot 2(1+x)^{-5}$$
$$\xrightarrow{D} f^{(6)}(x) = -5\cdot 4\cdot 3\cdot 2(1+x)^{-6} \xrightarrow{D} \cdots$$
$$f^{(n)}(x) = (-1)^{n+1}(n-1)!(1+x)^{-n} \text{ for } n=1,2,3,\ldots$$

so $f^{(n+1)}(x) = (-1)^{n+2} \cdot n!\,(1+x)^{-(n+1)}$; for x in $[0, 0.1]$ the Taylor Error

$$|R_{n+1}(x;0)| = \left| \frac{f^{(n+1)}(c)}{(n+1)!}(x-0)^{n+1} \right|$$

$$= \left| \frac{(-1)^{n+2} \cdot n!\,(1+c)^{-(n+1)}}{(n+1)!} \cdot x^{n+1} \right|$$

$$= \frac{|x|^{n+1}}{n+1} \cdot \frac{1}{|1+c|^{n+1}} \leq \frac{(0.1)^{n+1}}{n+1} \cdot \frac{1}{|1+0|^{n+1}}$$

$$= \boxed{\frac{(0.1)^{n+1}}{n+1} < 0.01}$$; guess and check:

n	$\frac{(0.1)^{n+1}}{n+1}$
0	0.1
1	$0.005 \leq 0.01$

So choose $\boxed{n=1}$ (or bigger)