First-Order Linear Solutions

1.) a.) $y' + 2\cdot y = 5 \quad \{\mu = e^{\int 2\,dx} = e^{2x}\} \rightarrow$

$e^{2x}y' + 2e^{2x}y = 5e^{2x} \rightarrow D[e^{2x}y] = 5e^{2x} \rightarrow$

$e^{2x}y = \int 5e^{2x}\,dx \rightarrow e^{2x}y = 5\cdot\frac{1}{2}e^{2x} + c \rightarrow$

$$\boxed{e^{2x}y = \frac{5}{2}e^{2x} + c}$$

b.) $y' + 1\cdot y = e^{3x} \quad \{\mu = e^{\int 1\,dx} = e^{x}\} \rightarrow$

$e^{x}y' + e^{x}y = e^{x}\cdot e^{3x} = e^{4x} \rightarrow$

$D[e^{x}y] = e^{4x} \rightarrow e^{x}y = \int e^{4x}\,dx \rightarrow$

$$\boxed{e^{x}y = \frac{1}{4}e^{4x} + c}$$

c.) $y' + 3x^2\cdot y = x^2 \quad \{\mu = e^{\int 3x^2\,dx} = e^{x^3}\} \rightarrow$

$e^{x^3}y' + 3x^2\cdot e^{x^3}\cdot y = x^2e^{x^3} \rightarrow$

$D[e^{x^3}y] = x^2e^{x^3} \rightarrow e^{x^3}y = \int x^2e^{x^3}\,dx \rightarrow$

$$\boxed{e^{x^3}y = \frac{1}{3}e^{x^3} + c}$$

d.) $x^2y' + xy = 1 \rightarrow y' + \frac{1}{x}\cdot y = \frac{1}{x^2}$

$\{\mu = e^{\int \frac{1}{x}\,dx} = e^{\ln x} = x\} \rightarrow$

$xy' + y = \frac{1}{x} \rightarrow D[xy] = \frac{1}{x} \rightarrow$

$$xy = \int \frac{1}{x}\,dx \rightarrow \boxed{xy = \ln|x| + c}$$

e.) $(1+x^2)y' + xy + x^3 + x = 0 \rightarrow$

$$(1+x^2)y' + xy = -x^3 - x \rightarrow$$

$$y' + \frac{x}{1+x^2}y = \frac{-x(x^2+1)}{1+x^2} = -x$$

$$\left\{ \mu = e^{\int \frac{x}{1+x^2}dx} = e^{\frac{1}{2}\ln(1+x^2)} = e^{\ln(1+x^2)^{\frac{1}{2}}} = \sqrt{1+x^2} \right\} \rightarrow$$

$$(1+x^2)^{\frac{1}{2}}y' + \frac{x}{(1+x^2)^{1/2}}y = -x(1+x^2)^{1/2} \rightarrow$$

$$D\left[(1+x^2)^{1/2}\cdot y\right] = -x(1+x^2)^{1/2} \rightarrow$$

$$(1+x^2)^{1/2}\cdot y = -\int x(1+x^2)^{1/2}dx \rightarrow$$

$$\boxed{(1+x^2)^{1/2}\cdot y = -\frac{2}{3}\cdot\frac{1}{2}(1+x^2)^{3/2} + c}$$

f.) $xy' + (1+x)y = e^{-x}\sin 2x \rightarrow$

$$y' + \left(\frac{1}{x}+1\right)y = \frac{1}{x}e^{-x}\sin 2x$$

$$\left\{ \mu = e^{\int \left(\frac{1}{x}+1\right)dx} = e^{\ln x + x} = e^{\ln x}\cdot e^{x} = xe^{x} \right\} \rightarrow$$

$xe^x y' + (1+x)e^x y = \sin 2x \rightarrow$

$D[xe^x y] = \sin 2x \rightarrow xe^x y = \int \sin 2x \, dx$

$\rightarrow \boxed{xe^x y = -\frac{1}{2}\cos 2x + c}$

g.) $y' - y = x \quad \{\mu = e^{\int -1 dx} = e^{-x}\} \rightarrow$

$e^{-x}y' - e^{-x}y = xe^{-x} \rightarrow D[e^{-x}y] = xe^{-x} \rightarrow$

$e^{-x}y = \int xe^{-x} dx$ (Let $u = x$, $dv = e^{-x}dx$

$\rightarrow du = 1 dx$, $v = -e^{-x}$)

$= -xe^{-x} - \int e^{-x} dx$

$= -xe^{-x} + -e^{-x} + c \rightarrow$

$\boxed{e^{-x}y = -xe^{-x} - e^{-x} + c}$

h.) $y' - 2y = xe^{2x} \quad \{\mu = e^{\int -2dx} = e^{-2x}\} \rightarrow$

$e^{-2x}y' - 2e^{-2x}y = x \rightarrow$

$D[e^{-2x}y] = x \rightarrow e^{-2x}y = \int x \, dx \rightarrow$

$e^{-2x}y = \frac{1}{2}x^2 + c$ and $x = 0$, $y = 2 \rightarrow$

$(1)(2) = 0 + c \rightarrow c = 2 \rightarrow \boxed{e^{-2x}y = \frac{1}{2}x^2 + 2}$

i.) $\cos x \cdot y' + \sin x \cdot y = 1 \rightarrow$

$$y' + \frac{\sin x}{\cos x}\, y = \frac{1}{\cos x}$$

$$\left\{ \mu = e^{\int \frac{\sin x}{\cos x}\,dx} = e^{-\ln(\cos x)}\right.$$

$$\left. = e^{\ln(\cos x)^{-1}} = \frac{1}{\cos x} \right\} \rightarrow$$

$$\frac{1}{\cos x} \cdot y' + \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} \cdot y = \frac{1}{\cos^2 x} \rightarrow$$

$$\sec x \cdot y' + \sec x \cdot \tan x \cdot y = \sec^2 x \rightarrow$$

$$D[\sec x \cdot y] = \sec^2 x \rightarrow$$

$$\sec x \cdot y = \int \sec^2 x\, dx \rightarrow$$

$$\boxed{\sec x \cdot y = \tan x + c}$$

j.) $y' + y = \dfrac{1 - e^{-2x}}{e^x + e^{-x}} \quad \left\{ \mu = e^{\int 1\,dx} = e^x \right\} \rightarrow$

$$e^x y' + e^x y = \frac{e^x - e^{-x}}{e^x + e^{-x}} \rightarrow$$

$$D[e^x y] = \frac{e^x - e^{-x}}{e^x + e^{-x}} \rightarrow$$

$$e^x y = \int \frac{e^x - e^{-x}}{e^x + e^{-x}}\, dx \rightarrow$$

$$\boxed{e^x y = \ln(e^x + e^{-x}) + c}$$

k.) $(1+x)y' - xy = x + x^2 = x(1+x) \rightarrow$

$$y' + \frac{-x}{1+x}\cdot y = \frac{x(\cancel{1+x})}{(\cancel{1+x})} = x$$

$\left\{ \mu = e^{\int \frac{-x}{1+x}dx} \right.$: let $u = 1+x$, $x = u-1$ and $du = dx$,

$$= e^{\int -\frac{(u-1)}{u}du} = e^{\int(-1+\frac{1}{u})du}$$

$$\left. = e^{-u+\ln u} = e^{-(1+x)}e^{\ln(1+x)} = \frac{1+x}{e^{1+x}} \right\} \rightarrow$$

$$\frac{1+x}{e^{1+x}}\cdot y' - \frac{x}{e^{1+x}}\cdot y = \frac{x+x^2}{e^{1+x}} \longrightarrow$$

$$D\left[\frac{1+x}{e^{1+x}}\cdot y\right] = (x+x^2)e^{-1-x} \longrightarrow$$

$$\frac{1+x}{e^{1+x}}\cdot y = \int (x+x^2)e^{-1-x}dx$$

$\{$ Let $u = x+x^2$, $dv = e^{-1-x}dx$

$\rightarrow du = (1+2x)dx$, $v = -e^{-1-x}\}$

$$= -(x+x^2)e^{-1-x} - -\int(1+2x)e^{-1-x}dx$$

$\{$ Let $u = 1+2x$, $dv = e^{-1-x}dx$

$\rightarrow du = 2\,dx$, $v = -e^{-1-x}\}$

$$= -(x+x^2)e^{-1-x}$$

$$+ \left[-(1+2x)e^{-1-x} - -2\int e^{-1-x}\,dx\right]$$

$$= -xe^{-1-x} - x^2e^{-1-x} - e^{-1-x} - 2xe^{-1-x}$$

$$+ 2\cdot(-1)e^{-1-x} + c$$

$$= -3e^{-1-x} - 3xe^{-1-x} - x^2e^{-1-x} + c \rightarrow$$

$$\boxed{\frac{1+x}{e^{1+x}}\cdot y = -3e^{-1-x} - 3xe^{-1-x} - x^2e^{-1-x}}$$

l.) $\cos^2 x\cdot\sin x\cdot y' + \cos^3 x\cdot y = 1 \rightarrow$

$$y' + \frac{\cos x}{\sin x}\cdot y = \frac{1}{\cos^2 x\cdot\sin x}$$

$$\left\{\mu = e^{\int \frac{\cos x}{\sin x}dx} = e^{\ln(\sin x)} = \sin x\right\} \rightarrow$$

$$\sin x\cdot y' + \cos x\cdot y = \frac{1}{\cos^2 x} = \sec^2 x \rightarrow$$

$$D[\sin x\cdot y] = \sec^2 x \rightarrow \sin x\cdot y = \int \sec^2 x\,dx$$

$$\rightarrow \boxed{\sin x\cdot y = \tan x + c}$$

Math 17B
Kouba
Worksheet 2

Let S represent the amount (in pounds) of salt in each tank at time t minutes. Find a formula for S for each of the following and then answer the particular questions.

1.) A solution containing 1/2 lb. of salt per gallon flows into a tank at the rate of 2 gal./min. and the well-stirred mixture flows out of the tank at the same rate. The tank initially holds 100 gallons of solution containing 5 lbs. of salt.

a.) How much salt is in the tank after 10 minutes ? after 1 hour ?
b.) How much salt do you expect to be in the tank as t gets infinitely large ?

2.) Pure water flows into a tank at the rate of 4 gal./min. and the well-stirred mixture flows out of the tank at the rate of 5 gal./min. The tank initially holds 200 gallons of water containing 50 lbs. of salt.

a.) How many gallons of solution are in the tank after 20 minutes ?
b.) How much salt is in the tank after 20 minutes ? after 2 hours ?
c.) How long will it take the tank to be empty ?

3.) A large tank holds 100 gallons of fluid in which 10 pounds of salt is dissolved. A mixture containg 1/2 lb. of salt per gallon flows into the tank at the rate of 6 gal./min. and the well-stirred mixture flows out of the tank at the rate of 4 gal./min.

a.) How many gallons of solution are in the tank after 10 minutes ? after 1 hour ?
b.) How much salt is in the tank after 10 minutes ? after 1 hour ?
c.) In how many minutes will the tank contain 40 pounds of salt ? (HINT: Use a calculator with an equation solver to solve for t or just estimate the solution by trial-and-error.)

4.) Beer containing 6% alcohol per gallon is pumped into a vat which initially contains 400 gallons of beer at 3% alcohol. Beer is pumped into the tank at the rate of 3 gal./min. and the well-stirred mixture is is pumped out of the tank at the rate of 4 gal./min.

a.) How many gallons of beer are in the vat after 10 minutes ? after 1 hour ? after 6 hours and 40 minutes ?

b.) What is the percentage of alcohol in the vat after 10 minutes ? after 1 hour ?

c.) When will the percentage of alcohol in the vat be 4% ? (HINT: Use a calculator with an equation solver to solve for t or just estimate the solution by trial-and-error.)

Worksheet 2

Solutions

Let S be lbs. of salt in tank at time t.

1.) $\frac{dS}{dt} = (\text{Rate In}) - (\text{Rate Out})$

$$= \left(\frac{\frac{1}{2}\text{ lb.}}{\text{gal.}}\right)\left(\frac{2\text{ gal.}}{\text{min.}}\right) - \left(\frac{S\text{ lbs.}}{100\text{ gal.}}\right)\left(\frac{2\text{ gal.}}{\text{min.}}\right) \rightarrow$$

$$\boxed{\frac{dS}{dt} = 1 - \frac{1}{50}S} \rightarrow \int \frac{1}{1-\frac{1}{50}S}\,dS = \int dt \rightarrow$$

$$-50\ln\left|1 - \frac{1}{50}S\right| = t + C \quad \text{and}$$

$t = 0$ min., $S = 5$ lbs. $\rightarrow$

$$-50\ln(9/10) = 0 + C \rightarrow C = -50\ln\left(\frac{9}{10}\right) \rightarrow$$

$$-50\ln\left(1 - \frac{1}{50}S\right) = t - 50\ln\left(\frac{9}{10}\right) \rightarrow$$

(since $S < 50$) $\nearrow$

$$\ln\left(1 - \frac{1}{50}S\right) = \frac{-1}{50}t + \ln(9/10) \rightarrow$$

$$e^{\ln\left(1-\frac{1}{50}S\right)} = e^{-\frac{1}{50}t + \ln(9/10)} \rightarrow$$

$$1 - \frac{1}{50}S = e^{-\frac{1}{50}t}e^{\ln(9/10)} \rightarrow$$

$$1 - \frac{1}{50}S = \frac{9}{10}e^{-\frac{1}{50}t} \rightarrow$$

$$\frac{1}{50}S = 1 - \frac{9}{10}e^{-\frac{1}{50}t} \rightarrow$$

$$\boxed{S = 50 - 45e^{-1/50\,t}}$$

a.) $t = 10$ min. $\rightarrow$ $S = 50 - 45e^{-1/5} \approx 13.2$ lbs.,
$t = 60$ min. $\rightarrow$ $S = 50 - 45e^{-6/5} \approx 36.4$ lbs.

b.) $\lim_{t\to\infty} S = \lim_{t\to\infty} \left(50 - 45e^{-\frac{1}{50}t}\right)$

$= 50 - 45(e^{-\infty}) = 50 - 45(0)$

$= 50$ lbs.

Let S be lbs. of salt in tank at time t

2.) $\frac{dS}{dt} = (\text{Rate In}) - (\text{Rate Out})$

$= \left(\frac{0 \text{ lbs.}}{\text{gal.}}\right)\left(\frac{4 \text{ gal.}}{\text{min.}}\right) - \left(\frac{S \text{ lbs.}}{200-t \text{ gal.}}\right)\left(\frac{5 \text{ gal.}}{\text{min.}}\right) \rightarrow$

$\boxed{\frac{dS}{dt} = \frac{-5S}{200-t}} \rightarrow \int \frac{1}{S}\, dS = \int \frac{-5}{200-t}\, dt \rightarrow$

$\ln S = 5 \ln(200-t) + c$; and
$t = 0$ min., $S = 50$ lbs. $\rightarrow$

$\ln 50 = 5 \ln 200 + c \rightarrow$

$c = \ln 50 - \ln 200^5 = \ln\left(\frac{50}{200^5}\right) \rightarrow$

$c = \ln\left(\frac{1}{6,400,000,000}\right) \rightarrow$

$\ln S = 5 \ln(200-t) + \ln\left(\frac{1}{6,400,000,000}\right) \rightarrow$

$e^{\ln S} = e^{\ln(200-t)^5 + \ln\left(\frac{1}{6,400,000,000}\right)} \rightarrow$

$S = e^{\ln(200-t)^5} e^{\ln\left(\frac{1}{6,400,000,000}\right)} \rightarrow$

$$\boxed{S = \frac{1}{6{,}400{,}000{,}000}(200-t)^5}$$

a.) Tank loses $5-4=1$ gal./min so in 20 minutes there are $200-20=180$ gal. of solution.

b.) $t=20$ min. $\rightarrow$ $S \approx 29.52$ lbs.,
$t=120$ min. $\rightarrow$ $S \approx 0.512$ lbs.

c.) (See a.)) The tank is empty in 200 minutes.

Let S be lbs. of salt in tank at time t.

3.) $$\frac{dS}{dt} = (\text{Rate In}) - (\text{Rate Out})$$

$$= \left(\frac{\frac{1}{2}\text{ lb.}}{\text{gal.}}\right)\left(\frac{6\text{ gal.}}{\text{min.}}\right) - \left(\frac{S\text{ lbs.}}{100+2t\text{ gal.}}\right)\left(\frac{4\text{ gal.}}{\text{min.}}\right) \rightarrow$$

$$\frac{dS}{dt} = 3 - \frac{4}{100+2t}\cdot S \longrightarrow$$

$$\boxed{\frac{dS}{dt} + \frac{4}{100+2t}\cdot S = 3}$$ (First-order Linear);

$$\left\{ \mu = e^{\int \frac{4}{100+2t}\,dt} = e^{4\cdot\frac{1}{2}\ln(100+2t)}\right.$$

$$= e^{2\ln(100+2t)} = e^{\ln(100+2t)^2}$$

$$\left. = (100+2t)^2. \right\} \longrightarrow$$

$$(100+2t)^2 \cdot \frac{dS}{dt} + 4(100+2t)\cdot S = 3(100+2t)^2 \rightarrow$$

$$D\left[(100+2t)^2 \cdot S\right] = 3(100+2t)^2 \rightarrow$$

$$(100+2t)^2 \cdot S = \int 3(100+2t)^2\,dt \rightarrow$$

$$(100+2t)^2 \cdot S = \frac{1}{2}(100+2t)^3 + C \rightarrow$$

$$\left\{ \begin{array}{l} t=0,\ S=10 \text{ lbs.} \rightarrow \\ 100^2(10) = \frac{1}{2}(100)^3 \rightarrow \\ C = -40(100)^2 = -400{,}000 \end{array} \right\} \rightarrow$$

$$(100+2t)^2 \cdot S = \frac{1}{2}(100+2t)^3 - 400{,}000 \rightarrow$$

$$\boxed{S = \frac{1}{2}(100+2t) - \frac{400{,}000}{(100+2t)^2}} \; ;$$

a.) The tank gains $6-4=2$ gal./min

i.) $t=10$ min : $100+2(10) = 120$ gal.

ii.) $t=60$ min : $100+2(60) = 220$ gal.

b.) i.) $t=10$ min :

$$S = \frac{1}{2}(120) - \frac{400{,}000}{(120)^2} \approx 32.22 \text{ lbs.}$$

ii.) $t=60$ min :

$$S = \frac{1}{2}(220) - \frac{400{,}000}{(220)^2} \approx 101.74 \text{ lbs.}$$

c.) $S = 40$ lbs :

$$40 = \frac{1}{2}(100+2t) - \frac{400{,}000}{(100+2t)^2} \rightarrow$$

$$40 = 50 + t - \frac{400{,}000}{(100+2t)^2} \rightarrow$$

$$\frac{400{,}000}{(100+2t)^2} = 10 + t \rightarrow$$ (using a calculator)

$$t \approx 14.23 \text{ min.}$$

4.) Let S be the gallons of alcohol in the tank at time t. Then

$$\frac{dS}{dt} = (\text{Rate In}) - (\text{Rate Out})$$

$$= \left(\frac{0.06 \text{ gal. alc.}}{1 \text{ gal. beer}}\right) \cdot \left(\frac{3 \text{ gal. beer}}{\text{min.}}\right)$$

$$- \left(\frac{S \text{ gal. alc.}}{400 - t \text{ gal. beer}}\right)\left(\frac{4 \text{ gal. beer}}{\text{min.}}\right)$$

$$\rightarrow \quad \frac{dS}{dt} = 0.18 - \frac{4}{400-t} \cdot S$$

$$\rightarrow \quad \frac{dS}{dt} + \frac{4}{400-t} \cdot S = 0.18$$

$$\rightarrow \quad \frac{dS}{dt} + \frac{-4}{t-400} \cdot S = 0.18$$ (First-order Linear)

$$\left\{ \mu = e^{\int \frac{-4}{t-400}\,dt} = e^{-4\ln(t-400)} = e^{\ln(t-400)^{-4}} = (t-400)^{-4} \right\} \rightarrow$$

$$(t-400)^{-4} \cdot \frac{dS}{dt} - 4(t-400)^{-5} \cdot S = 0.18\,(t-400)^{-4} \rightarrow$$

$$D\left[(t-400)^{-4} \cdot S\right] = 0.18\,(t-400)^{-4} \rightarrow$$

$$(t-400)^{-4} \cdot S = \int 0.18\,(t-400)^{-4}\,dt \rightarrow$$

$$(t-400)^{-4} \cdot S = (0.18)\left(\frac{-1}{3}\right)(t-400)^{-3} + C$$

$$\left\{ t=0,\ S = (3\%)(400) = 12 \text{ gal.} \rightarrow \frac{12}{(400)^4} = \frac{0.06}{(400)^3} + C \rightarrow C = \frac{-12}{400^4} \right\} \rightarrow$$

$$(t-400)^{-4} S = -0.06\,(t-400)^{-3} - \frac{12}{400^4} \rightarrow$$

$$S = -0.06\,(t-400) - \frac{12}{400^4}(t-400)^4 \rightarrow$$

$$\boxed{S = 24 - 0.06t - \frac{12}{400^4}(t-400)^4}\ ;$$

a.) The vat loses $4-3=1$ gal./min.

i.) $t=10$ min.: $400-10(1)=390$ gal.

ii.) $t=60$ min.: $400-60(1)=340$ gal.

iii.) $t=400$ min: $400-400(1)=0$ gal.

b.) i.) $t=10$ min.:

$S=24-0.6-\frac{12}{400^4}(-390)^4 \approx 12.56$ gal. alc.,

390 gal. beer so % of alcohol is

$\frac{12.56}{390} \approx 3.22\ \%$

ii.) $t=60$ min.:

$S=24-3.6-\frac{12}{400^4}(-340)^4 \approx 14.14$ gal. alc.,

340 gal. beer so % of alcohol is

$\frac{14.14}{340} \approx 4.16\ \%$

c.) $4\% = \frac{\text{gal. alc.}}{\text{gal. beer}} = \frac{24-0.06t-\frac{12}{400^4}(t-400)^4}{400-t}$

$\rightarrow 16-0.04t = 24-0.06t-\frac{12}{400^4}(t-400)^4$

$\rightarrow \frac{12}{400^4}(t-400)^4 = 8-0.02t \rightarrow$ (using a calculator)

$t \approx 50.57$ min.