# Section 9.2

1.) $A - B + 2C = \begin{bmatrix} -1 & 2 \\ 0 & -3 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 2 & 4 \end{bmatrix} + 2\begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}$

$= \begin{bmatrix} -1 & 2 \\ 0 & -3 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ -2 & -4 \end{bmatrix} + \begin{bmatrix} 2 & -4 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 0 & -9 \end{bmatrix}$

7.) $2A + 3B - C$

$= 2\begin{bmatrix} 1 & 3 & -1 \\ 2 & 4 & 1 \\ 0 & -2 & 2 \end{bmatrix} + 3\begin{bmatrix} 5 & -1 & 4 \\ 2 & 0 & 1 \\ 1 & -3 & -3 \end{bmatrix} - \begin{bmatrix} -2 & 0 & 4 \\ 1 & -3 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

$= \begin{bmatrix} 2 & 6 & -2 \\ 4 & 8 & 2 \\ 0 & -4 & 4 \end{bmatrix} + \begin{bmatrix} 15 & -3 & 12 \\ 6 & 0 & 3 \\ 3 & -9 & -9 \end{bmatrix} + \begin{bmatrix} 2 & 0 & -4 \\ -1 & 3 & -1 \\ 0 & 0 & -2 \end{bmatrix}$

$= \begin{bmatrix} 19 & 3 & 6 \\ 9 & 11 & 4 \\ 3 & -13 & -7 \end{bmatrix}$

15.) $A = \begin{bmatrix} -1 & 0 & 3 \\ 2 & 1 & -4 \end{bmatrix} \longrightarrow A' = \begin{bmatrix} -1 & 2 \\ 0 & 1 \\ 3 & -4 \end{bmatrix}$

21.) a.) $AB = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & -3 \\ 0 & 5 \end{bmatrix}$

b.) $BA = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 6 \\ 2 & 2 \end{bmatrix}$

24.) $AB = \begin{bmatrix} -2 & -3 \\ 0 & 5 \end{bmatrix}$, $BC = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & -3 \end{bmatrix}$,

$(AB)C = \begin{bmatrix} -2 & -3 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 0 & -5 \end{bmatrix}$, and

$$A(BC) = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ -1 & -3 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 0 & -5 \end{bmatrix}$$

26.) $B + C = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ -1 & 0 \end{bmatrix}$,

$AC = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 1 & 0 \end{bmatrix}$,

$A(B+C) = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -3 & -5 \\ 1 & 5 \end{bmatrix}$,

$AB + AC = \begin{bmatrix} -2 & -3 \\ 0 & 5 \end{bmatrix} + \begin{bmatrix} -1 & -2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -3 & -5 \\ 1 & 5 \end{bmatrix}$

27.) $AB = (3 \times 4)(4 \times 2) = (3 \times 2)$

29.) a.) $BD' = (1 \times 3)(3 \times 4) = 1 \times 4$

b.) $D'A = (3 \times 4)(4 \times 3) = 3 \times 3$

c.) $ACB = (AC)B$

$= ((4 \times 3)(3 \times 1))(1 \times 3) = (4 \times 1)(1 \times 3) = 4 \times 3$

34.) $AB = \begin{bmatrix} 1 & -1 \\ 3 & 0 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 2 & 4 & 1 \\ 6 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -4 & 4 & 1 \\ 6 & 12 & 3 \\ 22 & 20 & 5 \end{bmatrix}$,

$A' = \begin{bmatrix} 1 & 3 & 5 \\ -1 & 0 & 2 \end{bmatrix}$, $B' = \begin{bmatrix} 2 & 6 \\ 4 & 0 \\ 1 & 0 \end{bmatrix}$,

$(AB)' = \begin{bmatrix} -4 & 6 & 22 \\ 4 & 12 & 20 \\ 1 & 3 & 5 \end{bmatrix}$, $B'A' = \begin{bmatrix} 2 & 6 \\ 4 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 3 & 5 \\ -1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -4 & 6 & 22 \\ 4 & 12 & 20 \\ 1 & 3 & 5 \end{bmatrix}$

35.) $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ a.) $B^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,
$B^3 = B^2 B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $B^4 = B^2 B^2$
$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B^5 = B^3 B^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

b.) $B^k = \begin{cases} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & \text{if } k \text{ is odd} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \text{if } k \text{ is even} \end{cases}$

37.) $A I_2 = \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}$,

$I_2 A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}$

39.) $\begin{bmatrix} 2 & 3 & -1 \\ 0 & 2 & 1 \\ 1 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$

42.) $\begin{bmatrix} 1 & -2 & 1 \\ -2 & 1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

43.) $\left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & -2 \end{array}\right]$

$\sim \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 2 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 2 \end{array}\right] \longrightarrow$

$A^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$.

45.) $\left[\begin{array}{cc|cc} -1 & 1 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & -1 & -1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & -1 & -1 & 0 \\ 0 & 5 & 2 & 1 \end{array}\right]$

$$\sim \left[\begin{array}{cc|cc} 1 & -1 & -1 & 0 \\ 0 & 1 & 2/5 & 1/5 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & -3/5 & 1/5 \\ 0 & 1 & 2/5 & 1/5 \end{array}\right] \longrightarrow$$

$$A^{-1} = \begin{bmatrix} -3/5 & 1/5 \\ 2/5 & 1/5 \end{bmatrix}$$

51.) a.) $A = \begin{bmatrix} -1 & 0 \\ 2 & -3 \end{bmatrix}$, $D = \begin{bmatrix} -2 \\ -5 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

so $AX = D$ : $\left.\begin{array}{r} -x_1 = -2 \\ 2x_1 - 3x_2 = -5 \end{array}\right\} \longrightarrow$

$x_1 = 2 \longrightarrow 2(2) - 3x_2 = -5 \longrightarrow$

$3x_2 = 9 \longrightarrow x_2 = 3$

b.) $\left[\begin{array}{cc|cc} -1 & 0 & 1 & 0 \\ 2 & -3 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & -1 & 0 \\ 2 & -3 & 0 & 1 \end{array}\right]$

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & -1 & 0 \\ 0 & -3 & 2 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & -1 & 0 \\ 0 & 1 & -2/3 & -1/3 \end{array}\right] \longrightarrow$$

$A^{-1} = \begin{bmatrix} -1 & 0 \\ -2/3 & -1/3 \end{bmatrix}$ ; then

$AX = D \rightarrow A^{-1}AX = A^{-1}D \rightarrow$

$I_2 X = A^{-1}D \longrightarrow$

$$X = A^{-1}D = \begin{bmatrix} -1 & 0 \\ -2/3 & -1/3 \end{bmatrix} \begin{bmatrix} -2 \\ -5 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \longrightarrow$$

$x_1 = 2,\ x_2 = 3$

53.) $\det \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} = (2)(3) - (-1)(1)$
$= 6 + 1 = 7 \neq 0$
so $A$ is invertible

55.) $\det \begin{bmatrix} 4 & -1 \\ 8 & -2 \end{bmatrix} = (4)(-2) - (-1)(8)$
$= -8 + 8 = 0$ so $A$ is NOT invertible

63.) $\det \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} = (1)(2) - (-1)(0) = 2 \neq 0$
so $A$ is invertible;

$\left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1/2 \end{array}\right]$

$\sim \left[\begin{array}{cc|cc} 1 & 0 & 1 & 1/2 \\ 0 & 1 & 0 & 1/2 \end{array}\right] \rightarrow A^{-1} = \begin{bmatrix} 1 & 1/2 \\ 0 & 1/2 \end{bmatrix}$; then

$AX = \vec{0} \rightarrow A^{-1}AX = \vec{0} \rightarrow I_2 X = \vec{0} \rightarrow$
$X = \vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow x_1 = 0, x_2 = 0.$

65.) $\det \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix} = (1)(3) - (1)(3) = 3 - 3 = 0$
so $C$ is NOT invertible; then
$AX = \vec{0} \rightarrow \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow$

$x_1 + 3x_2 = 0 \rightarrow x_1 = -3x_2$; let $\boxed{x_2 = t}$
any real #, then $\boxed{x_1 = -3t}$

67.) $$\left[\begin{array}{ccc|ccc} 2 & -1 & -1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ -1 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 0 & 0 & -1 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 2 & -1 & -1 & 1 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 0 & 0 & -1 \\ 0 & 3 & -1 & 0 & 1 & 2 \\ 0 & 1 & -3 & 1 & 0 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 0 & 0 & -1 \\ 0 & 1 & -3 & 1 & 0 & 2 \\ 0 & 3 & -1 & 0 & 1 & 2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 1 \\ 0 & 1 & -3 & 1 & 0 & 2 \\ 0 & 0 & 8 & -3 & 1 & -4 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 1 \\ 0 & 1 & -3 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3/8 & 1/8 & -1/2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/4 & 1/4 & 0 \\ 0 & 1 & 0 & -1/8 & 3/8 & 1/2 \\ 0 & 0 & 1 & -3/8 & 1/8 & -1/2 \end{array}\right] \longrightarrow A^{-1} = \begin{bmatrix} 1/4 & 1/4 & 0 \\ -1/8 & 3/8 & 1/2 \\ -3/8 & 1/8 & -1/2 \end{bmatrix}$$

69.) $$\left[\begin{array}{ccc|ccc} -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & 1 & 0 \\ -1 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/2 & 0 \\ 0 & 1 & 3 & -1 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/2 & 0 \\ 0 & 0 & 3 & -1 & 1/2 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/2 & 0 \\ 0 & 0 & 1 & -1/3 & 1/6 & 1/3 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2/3 & -1/6 & -1/3 \\ 0 & 1 & 0 & 0 & -1/2 & 0 \\ 0 & 0 & 1 & -1/3 & 1/6 & 1/3 \end{array}\right] \longrightarrow A^{-1} = \begin{bmatrix} -2/3 & -1/6 & -1/3 \\ 0 & -1/2 & 0 \\ -1/3 & 1/6 & 1/3 \end{bmatrix}$$

71.) $$\begin{bmatrix} N_0(t+1) \\ N_1(t+1) \\ N_2(t+1) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 3.2 & 1.7 \\ 0.2 & 0 & 0 \\ 0 & 0.7 & 0 \end{bmatrix}}_{L} \begin{bmatrix} N_0(t) \\ N_1(t) \\ N_2(t) \end{bmatrix}$$

$$N(0) = \begin{bmatrix} 2000 \\ 800 \\ 200 \end{bmatrix},$$

$$L^2 = \begin{bmatrix} 0 & 3.2 & 1.7 \\ 0.2 & 0 & 0 \\ 0 & 0.7 & 0 \end{bmatrix} \begin{bmatrix} 0 & 3.2 & 1.7 \\ 0.2 & 0 & 0 \\ 0 & 0.7 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.64 & 1.19 & 0 \\ 0 & 0.64 & 0.34 \\ 0.14 & 0 & 0 \end{bmatrix}, \quad \text{then}$$

$$N(2) = L^2 N(0) = \begin{bmatrix} 0.64 & 1.19 & 0 \\ 0 & 0.64 & 0.34 \\ 0.14 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2000 \\ 800 \\ 200 \end{bmatrix}$$

$$= \begin{bmatrix} 2232 \\ 580 \\ 280 \end{bmatrix}$$

73.) $$\begin{bmatrix} N_0(t+1) \\ N_1(t+1) \\ N_2(t+1) \\ N_3(t+1) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 4.6 & 3.7 \\ 0.7 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \end{bmatrix}}_{L} \begin{bmatrix} N_0(t) \\ N_1(t) \\ N_2(t) \\ N_3(t) \end{bmatrix},$$

$$N(0) = \begin{bmatrix} 1500 \\ 500 \\ 250 \\ 50 \end{bmatrix},$$

$$L^2 = \begin{bmatrix} 0 & 0 & 4.6 & 3.7 \\ 0.7 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 4.6 & 3.7 \\ 0.7 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2.3 & 0.37 & 0 \\ 0 & 0 & 3.22 & 2.59 \\ 0.35 & 0 & 0 & 0 \\ 0 & 0.05 & 0 & 0 \end{bmatrix}, \quad \text{then}$$

$$N(2) = L^2 \cdot N(0) = \begin{bmatrix} 0 & 2.3 & 0.37 & 0 \\ 0 & 0 & 3.22 & 2.59 \\ 0.35 & 0 & 0 & 0 \\ 0 & 0.05 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1500 \\ 500 \\ 250 \\ 50 \end{bmatrix}$$

$$= \begin{bmatrix} 1243 \\ 935 \\ 525 \\ 25 \end{bmatrix}$$

75.) $L = \begin{bmatrix} 2 & 3 & 2 & 1 \\ 0.4 & 0 & 0 & 0 \\ 0 & 0.6 & 0 & 0 \\ 0 & 0 & 0.8 & 0 \end{bmatrix}$

a.) 4 age classes: $N(t) = \begin{bmatrix} N_0(t) \\ N_1(t) \\ N_2(t) \\ N_3(t) \end{bmatrix}$

b.) $0.6 = 60\%$ of $N_1(t)$ survive to end of next season

c.) 2 is average female offspring of 2 year old female

77.) $L = \begin{bmatrix} 1 & 2.5 & 3 & 1.5 \\ 0.9 & 0 & 0 & 0 \\ 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0.2 & 0 \end{bmatrix}$

a.) 4 age classes : $N(t) = \begin{bmatrix} N_0(t) \\ N_1(t) \\ N_2(t) \\ N_3(t) \end{bmatrix}$

b.) $0.2 = 20\%$ of $N_2(t)$ survive to end of ~~nex~~ next season

c.) 2.5 is average female offspring of 1 year old female