Section 7.3

1.)
$$\begin{array}{r|l} & 2x+1 \\ \hline x+2 & 2x^2+5x-1 \\ & -(2x^2+4x) \\ \hline & x-1 \\ & -(x+2) \\ \hline & -3 \end{array}$$
so $\dfrac{2x^2+5x-1}{x+2} = 2x+1+\dfrac{-3}{x+2}$

4.)
$$\begin{array}{r|l} & x-4 \\ \hline x^2+x+3 & x^3-3x^2-15 \\ & -(x^3+x^2+3x) \\ \hline & -4x^2-3x-15 \\ & -(-4x^2-4x-12) \\ \hline & x-3 \end{array}$$
so $\dfrac{x^3-3x^2-15}{x^2+x+3} = x-4+\dfrac{x-3}{x^2+x+3}$

5.) $\dfrac{2x-3}{x(x+1)} = \dfrac{A}{x}+\dfrac{B}{x+1} = \dfrac{A(x+1)+Bx}{x(x+1)} \rightarrow$

$2x-3 = A(x+1)+Bx$:

Let $x=0$: $-3 = A+0 \rightarrow A=-3$

Let $x=-1$: $-5 = 0-B \rightarrow B=5$, so that

$\dfrac{2x-3}{x(x+1)} = \dfrac{-3}{x}+\dfrac{5}{x+1}$

9.) $\dfrac{5x-1}{x^2-1} = \dfrac{5x-1}{(x-1)(x+1)} = \dfrac{A}{x-1}+\dfrac{B}{x+1}$

$= \dfrac{A(x+1)+B(x-1)}{(x-1)(x+1)} \rightarrow 5x-1 = A(x+1)+B(x-1)$:

Let $x=1$: $4 = 2A+0 \rightarrow A=2$

Let $x=-1$: $-6 = 0 - 2B \to B = 3$, so that

$$\frac{5x-1}{x^2-1} = \frac{2}{x-1} + \frac{3}{x+1}$$

11.) $$\frac{4x+1}{x^2-3x-10} = \frac{4x+1}{(x-5)(x+2)} = \frac{A}{x-5} + \frac{B}{x+2}$$

$$= \frac{A(x+2) + B(x-5)}{(x-5)(x+2)} \to$$

$4x+1 = A(x+2) + B(x-5)$:

Let $x=5$: $21 = 7A + 0 \to A = 3$

Let $x=-2$: $-7 = 0 - 7B \to B = 1$, so that

$$\frac{4x+1}{x^2-3x-10} = \frac{3}{x-5} + \frac{1}{x+2}$$

13.) $$\int \frac{1}{x(x-2)}\,dx = \int \left[\frac{A}{x} + \frac{B}{x-2}\right] dx$$

($1 = A(x-2) + Bx$:

Let $x=0$: $1 = -2A + 0 \to A = -1/2$

Let $x=2$: $1 = 0 + 2B \to B = 1/2$)

$$= \int \left[\frac{-1/2}{x} + \frac{1/2}{x-2}\right] dx = \frac{-1}{2}\ln|x| + \frac{1}{2}\ln|x-2| + C$$

17.) $$\int \frac{x^2-2x-2}{x^2(x+2)}\,dx = \int \left[\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2}\right] dx$$

($x^2-2x-2 = Ax(x+2) + B(x+2) + Cx^2$:

Let $x=0$: $-2 = 0 + 2B + 0 \to B = -1$

Let $x=-2$: $6 = 0 + 0 + 4C \to C = 3/2$

Let $x=-1$: $1 = -A + (-1)(1) + (3/2)(1) \to$

$A = -2 + 3/2 = \frac{-4}{2} + \frac{3}{2} = \frac{-1}{2}$)

$= \int \left[\frac{-1/2}{x} + \frac{-1}{x^2} + \frac{3/2}{x+2} \right] dx$

$= \frac{-1}{2} \ln|x| + \frac{1}{x} + \frac{3}{2} \ln|x+2| + C$

19.) $\int \frac{x^3 - x^2 + x - 4}{(x^2+1)(x^2+4)} dx = \int \left[\frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+4} \right] dx$

($x^3 - x^2 + x - 4 = (Ax+B)(x^2+4) + (Cx+D)(x^2+1)$:

Let $x = i$: $-i - (-1) + i - 4 = (Ai+B)(-1+4) + (Ci+D)(-1+1) \rightarrow$

$-3 = 3Ai + 3B \rightarrow (-3) + (0)i = (3B) + (3A)i \rightarrow$

$-3 = 3B \rightarrow \boxed{B = -1}$ and $0 = 3A \rightarrow \boxed{A = 0}$;

Let $x = 2i$: $-8i - (-4) + 2i - 4 = (2Ai + B)(-4+4)$

$+ (2Ci + D)(-4+1) \rightarrow -6i = -6Ci - 3D \rightarrow$

$(-6)i + (0) = (-6C)i + (-3D) \rightarrow -6 = -6C \rightarrow \boxed{C = 1}$

and $0 = -3D \rightarrow \boxed{D = 0}$)

$= \int \left[\frac{-1}{x^2+1} + \frac{x}{x^2+4} \right] dx = -\arctan x + \frac{1}{2} \ln|x^2+4| + C$

22.) $\int \frac{3x^2 + 4x + 3}{(x^2+1)^2} dx = \int \left[\frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} \right] dx$

($3x^2 + 4x + 3 = (Ax+B)(x^2+1) + (Cx+D)$

Let $x = i$: $3(-1) + 4i + 3 = (Ai+B)(-1+1) + (Ci+D) \rightarrow$

$4i = Ci + D \rightarrow (4)i + (0) = (C)i + (D) \rightarrow$

$C = 4$, $D = 0$,

Let $x = 0$: $3 = B + (0+0) \rightarrow B = 3$,

Let $x = 1$: $10 = (A+3)(2) + (4+0) \rightarrow 6 = (A+3)2$

$\rightarrow 3 = A + 3 \rightarrow \boxed{A = 0}$)

$$= \int \left[\frac{3}{x^2+1} + \frac{4x}{(x^2+1)^2} \right] dx$$

$$= 3 \arctan x + {}^{-}2(x^2+1)^{-1} + C$$

23.) $\displaystyle \int \frac{1}{x^2-2x+2}\,dx = \int \frac{1}{(x^2-2x+1)+1}\,dx$

$$= \int \frac{1}{(x-1)^2+1}\,dx = \arctan(x-1) + C$$

24.) $\displaystyle \int \frac{1}{x^2+4x+5}\,dx = \int \frac{1}{(x^2+4x+4)+1}\,dx$

$$= \int \frac{1}{(x+2)^2+1}\,dx = \arctan(x+2) + C$$

25.) $\displaystyle \int \frac{1}{x^2-4x+13}\,dx = \int \frac{1}{(x^2-4x+4)+9}\,dx$

$$= \int \frac{1}{(x-2)^2+3^2}\,dx = \frac{1}{3} \arctan\left(\frac{x-2}{3}\right) + C$$

27.) $\int \frac{1}{(x-3)(x+2)}\,dx = \int \left[\frac{A}{x-3} + \frac{B}{x+2}\right] dx$

$\Big(A(x+2) + B(x-3) = 1$: then

let $x = -2$: $\quad -5B = 1 \rightarrow \quad B = -1/5$,

let $x = 3$: $\quad 5A = 1 \rightarrow \quad A = 1/5 \Big)$

$= \int \left[\frac{\frac{1}{5}}{x-3} + \frac{\frac{-1}{5}}{x+2}\right] dx = \frac{1}{5}\ln|x-3| - \frac{1}{5}\ln|x+2| + c$

29.) $\int \frac{1}{x^2-9}\,dx = \int \frac{1}{(x-3)(x+3)}\,dx = \int \left[\frac{A}{x-3} + \frac{B}{x+3}\right] dx$

$\Big(A(x+3) + B(x-3) = 1$: then

let $x = 3$: $\quad 6A = 1 \rightarrow A = 1/6$,

let $x = -3$: $\quad -6B = 1 \rightarrow \quad B = -1/6 \Big)$

$= \int \left[\frac{\frac{1}{6}}{x-3} + \frac{\frac{-1}{6}}{x+3}\right] dx = \frac{1}{6}\ln|x-3| - \frac{1}{6}\ln|x+3| + c$

30.) $\int \frac{1}{x^2+9}\,dx = \int \frac{1}{x^2+3^2}\,dx = \frac{1}{3}\arctan\frac{x}{3} + c$

31.) $\int \frac{1}{x^2-x-2}\,dx = \int \frac{1}{(x-2)(x+1)}\,dx = \int \left[\frac{A}{x-2} + \frac{B}{x+1}\right] dx$

$\Big(A(x+1) + B(x-2) = 1$: then

let $x = 2$: $\quad 3A = 1 \rightarrow A = 1/3$,

let $x = -1$: $\quad -3B = 1 \rightarrow \quad B = -1/3 \Big)$

$= \int \left[\frac{\frac{1}{3}}{x-2} + \frac{-1/3}{x+1}\right] dx = \frac{1}{3}\ln|x-2| - \frac{1}{3}\ln|x+1| + c$

32.) $\int \frac{1}{x^2-x+2}\,dx = \int \frac{1}{(x^2-x+\frac{1}{4})+2-\frac{1}{4}}\,dx$

complete the square

$$= \int \frac{1}{(x-\frac{1}{2})^2 + \frac{7}{4}}\,dx = \int \frac{1}{(x-\frac{1}{2})^2 + \left(\frac{\sqrt{7}}{2}\right)^2}\,dx$$

(Let $u = x - \frac{1}{2} \xrightarrow{D} du = 1\,dx$)

$$= \int \frac{1}{u^2 + \left(\frac{\sqrt{7}}{2}\right)^2}\,du = \frac{1}{\sqrt{7}/2} \arctan\left(\frac{u}{\sqrt{7}/2}\right) + c$$

$$= \frac{2}{\sqrt{7}} \arctan \frac{2}{\sqrt{7}}(x - 1/2) + c$$

34.) $\displaystyle\int \frac{x^3+1}{x^2+3}\,dx$

$$\begin{array}{r} x \\ x^2+3 \,\overline{)\, x^3 + 1 \quad\;} \\ -(x^3 + 3x) \\ \hline -3x + 1 \end{array}$$

$$= \int \left[x + \frac{-3x+1}{x^2+3}\right] dx$$

$$= \frac{1}{2}x^2 + \int \left[\frac{-3x}{x^2+3} + \frac{1}{x^2+3}\right] dx$$

$$= \frac{1}{2}x^2 + \int \frac{1}{x^2+(\sqrt{3})^2}\,dx - 3\int \frac{x}{x^2+3}\,dx$$

(Let $u = x^2 + 3 \xrightarrow{D} du = 2x\,dx$
$\longrightarrow \frac{1}{2}du = x\,dx$)

$$= \frac{1}{2}x^2 + \frac{1}{\sqrt{3}} \arctan \frac{x}{\sqrt{3}} - 3 \cdot \frac{1}{2} \int \frac{1}{u}\,du$$

$$= \frac{1}{2}x^2 + \frac{1}{\sqrt{3}} \arctan \frac{x}{\sqrt{3}} - \frac{3}{2} \ln|u| + c$$

$$= \frac{1}{2}x^2 + \frac{1}{\sqrt{3}} \arctan \frac{x}{\sqrt{3}} - \frac{3}{2} \ln|x^2+3| + c$$

35.) $\displaystyle\int \frac{x^2+4}{x^2-4}\,dx$

$$\begin{array}{r} 1 \\ x^2-4 \,\overline{)\, x^2 + 4 \quad} \\ -(x^2 - 4) \\ \hline 8 \end{array}$$

$$= \int \left[1 + \frac{8}{x^2-4}\right] dx$$

$$= x + \int \frac{8}{(x-2)(x+2)}\,dx = x + \int \left[\frac{A}{x-2} + \frac{B}{x+2}\right] dx$$

$(A(x+2) + B(x-2) = 8$: then
let $x = 2$: $4A = 8 \rightarrow A = 2$,
let $x = -2$: $-4B = 8 \rightarrow B = -2$)

$$= x + \int \left[\frac{2}{x-2} + \frac{-2}{x+2}\right] dx$$

$$= x + 2\ln|x-2| - 2\ln|x+2| + c$$

38.) $\int_3^5 \frac{x}{x-1}\,dx$ (Let $u = x - 1$ then $x = u+1$
$\xrightarrow{D} du = 1\,dx$; $x: 3 \to 5$ so $u: 2 \to 4$)

$$= \int_2^4 \frac{u+1}{u}\,du = \int_2^4 \left(1 + \frac{1}{u}\right) du = \left(u + \ln|u|\right)\Big|_2^4$$

$$= (4 + \ln 4) - (2 + \ln 2) = 4 + \ln 4 - 2 - \ln 2$$

$$= 2 + \ln(4/2) = 2 + \ln 2$$

43.) $\int_0^1 \arctan x\,dx$ (Let $u = \arctan x$, $dv = dx$
$\rightarrow du = \frac{1}{1+x^2}\,dx$, $v = x$)

$$= x\arctan x\,\Big|_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx$$

(Let $u = 1 + x^2 \xrightarrow{D} du = 2x\,dx \rightarrow \frac{1}{2}du = x\,dx$;
$x: 0 \to 1$ so $u: 1 \to 2$)

$$= 1\cdot\arctan 1 - 0\cdot\arctan 0 - \frac{1}{2}\int_1^2 \frac{1}{u}\,du$$

$$= 1\cdot\frac{\pi}{4} - 0\cdot 0 - \frac{1}{2}\ln|u|\,\Big|_1^2$$

$$= \frac{\pi}{4} - \left(\frac{1}{2}\ln 2 - \frac{1}{2}\cancelto{0}{\ln 1}\right) = \frac{\pi}{4} - \frac{1}{2}\ln 2$$

45.) $\int \frac{1}{(x+1)^2 x}\,dx = \int \left[\frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x}\right] dx$

($A(x+1)x + Bx + C(x+1)^2 = 1$: then

let $x=0$: $\boxed{C=1}$,

let $x=-1$: $-B = 1 \rightarrow \boxed{B=-1}$,

let $x=1$: $2A + (-1) + (1)(2)^2 = 1 \rightarrow$

$2A - 1 + 4 = 1 \rightarrow 2A = -2 \rightarrow \boxed{A=-1}$)

$= \int \left[\frac{-1}{x+1} + \frac{-1}{(x+1)^2} + \frac{1}{x}\right] dx$

$= \int \left[\frac{-1}{x+1} - (x+1)^{-2} + \frac{1}{x}\right] dx$

$= -\ln|x+1| - \frac{(x+1)^{-1}}{-1} + \ln|x| + C$

46.) $\int \frac{1}{x^2(x-1)^2}\,dx = \int \left[\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2}\right] dx$

($Ax(x-1)^2 + B(x-1)^2 + Cx^2(x-1) + Dx^2 = 1$: then

let $x=0$: $\boxed{B=1}$,

let $x=1$: $\boxed{D=1}$,

let $x=2$: $2A + 1 + 4C + 4 = 1 \rightarrow$

$2A + 4C = -4 \rightarrow \boxed{A+2C=-2}$,

let $x=-1$: $-4A + 4 - 2C + 1 = 1 \rightarrow$

$2C + 4A = 4 \rightarrow \boxed{C+2A=2}$; then

$C = 2 - 2A \rightarrow$ (sub) $A + 2(2-2A) = -2 \rightarrow$

$A + 4 - 4A = -2 \rightarrow 6 = 3A \rightarrow \boxed{A=2}$

and $\boxed{C=-2}$)

$$= \int \left[\frac{2}{x} + \frac{1}{x^2} + \frac{-2}{x-1} + \frac{1}{(x-1)^2} \right] dx$$

$$= 2\ln|x| + \frac{-1}{x} - 2\ln|x-1| + \frac{-1}{x-1} + C$$

51.) $\int \frac{1}{x^2(x^2+1)}\,dx = \int \left[\frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1} \right] dx$

($Ax(x^2+1) + B(x^2+1) + (Cx+D)x^2 = 1$: then

let $x=0$: $\boxed{B=1}$,

let $x=i$: $(i^2=-1)$, $(Ci+D)(-1)=1 \rightarrow$

$(-C)i + (-D) = (0)i + (1) \rightarrow \boxed{C=0}$, $\boxed{D=-1}$,

let $x=1$: $2A+2-1=1 \rightarrow 2A=0 \rightarrow \boxed{A=0}$)

$$= \int \left[\frac{1}{x^2} + \frac{-1}{x^2+1} \right] dx = \frac{-1}{x} - \arctan x + C$$